Polylogarithm function

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Background

Today I am trying to do an integral

$$\int_0^{2\pi} \frac{t}{1+e^t}\,dt. \tag{1}$$

I do not know how to do it so I tried to use Mathematica, which gives me

$$\frac{23\pi^2}{12} - 2\pi \mathit{log}(1+e^{2\pi}) - \mathit{PolyLog}[2, -e^{2\pi}]. \tag{2}$$

I do not understand what is "PolyLog".

PolyLog

I checked Wikipedia [2] and learned that it is called a Polylogrithm function, which is a nonelementary functions (elementary functions are functions with finite number of arithmic operations). It is defined as, according to Wikipedia

$$Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}. \tag{3}$$

Thus, the one I encountered will be an infinite sum such as

$$PolyLog[2, -e^{-2\pi}] = \sum_{k=1}^{\infty} \frac{(-e^{-2\pi})^k}{k^2}. \tag{4}$$

In General

I did not get into the details of why the integral turned out to be a Polylogrithm, according to the definition. However, what is useful is that, in general,

$$Li_s(z) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} dt}{e^t/z - 1} \tag{5}$$

where $\Gamma(s)$ is called Gamma function, defined as $\Gamma(n) = (n-1)!$ [1].

References

Wikipedia contributors. *Gamma function — Wikipedia, The Free Encyclopedia*. [Online; accessed 25-November-2018]. 2018. URL: https://en.wikipedia.org/w/index.php?title=Gamma_function&oldid=870308554.

Wikipedia contributors. *Polylogarithm — Wikipedia, The Free Encyclopedia*. [Online; accessed 25-November-2018]. 2018. URL: https://en.wikipedia.org/w/index.php?title=Polylogarithm&oldid=862627827.